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An alternative method to solve the hadronic cosmic-ray diffusion equations: the muon and neutrino fluxes

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Abstract. The integro-differential equations which describe diffusion of the hadronic component in the atmosphere are exactly solved by means of the successive approximation method. The numbers of muons and neutrinos produced are derived from the hadron fluxes obtained as the solution of the above-mentioned equations. The primary cosmic ray spectrum used in our calculation is presented in a general form $G(E)$.

1. Introduction

Several authors have studied analytically the diffusion equations of hadrons in the atmosphere. Ohsawa [1] solved these equations applying a Laplace transformation for the depth and a Mellin transformation for the energy. Mackewon, Sidhanta and others [2] solved these equations applying only the Mellin transformation on the variable E (energy).

If we use the method of Mellin's transform we obtain a real solution represented by a contour integral in the complex domain and, in a few particular cases, this integral can be evaluated exactly; in the general case, however, we must use some approximate method to estimate it, for example, the saddle point method.

In this paper we used the successive approximation method to solve these diffusion equations with a boundary condition, $N(O, E)=G(E)$, where $G(E)$ is a continuous, positive and bounded function, representing the primary cosmic ray energy spectrum.

We obtain the differential fluxes of hadrons, muons and neutrinos in an exact and compact form, and for the particular case $N(O, E)=N_0E^{-(\gamma+1)}$ our solutions result in the generally used expressions.

2. Nucleon diffusion equation in the atmosphere

The diffusion of nucleons in the atmosphere can be represented by the one-dimensional integro-differential equations

$$\frac{\partial N(x, E)}{\partial x}=-\frac{N(x, E)}{\lambda}+\int_E^\infty \frac{N(x, E')}{\lambda} f(E, E')\,\mathrm{d}E' \tag{2.1}$$

where $N(x, E)$ is the differential nucleon flux at depth x and at energies between E and $E+dE$, and λ is the nucleon interaction mean free path in the atmosphere. E', E are respectively the primary and secondary nucleon energy, $f(E, E')$ are the energy distributions of the secondary nucleons.

These functions are homogeneous of order (-1), because we assume the scaling hypothesis, so that the equations (2.1) take the form

$$\frac{\partial N(x,E)}{\partial x} = -\frac{N(x,E)}{\lambda} + \int_0^1 \frac{N(x, E/\eta)}{\lambda} f(\eta)\, d\eta/\eta \tag{2.2}$$

with the boundary condition

$$N(O, E) = G(E)$$

where $G(E)\, dE$ is the differential energy spectrum of the nucleons at the top of the atmosphere. This function is supposed to be continuous, positive and bounded ($G(E) \leqslant M$) in the interval $0 < E_{\min} \leqslant E < \infty$. The existence of the integral $\int_E^\infty G(E)\, dE$, for $E \geqslant E_{\min}$, must also be stated because it represents the primary integral spectrum.

Let us put $N(x, E) = e^{-x/\lambda} y(x, E)$, so that the equation (2.2) and the respective initial condition become

$$\frac{\partial y(x,E)}{\partial x} = \frac{1}{\lambda} \int_0^1 y(x, E/\eta) f(\eta)\, d\eta/\eta \tag{2.3}$$

with $y(0, E) = G(E)$.

Now we make the following successive approximations

$$y_0(x, E) = G(E)$$

$$y_n(x, E) = G(E) + \frac{1}{\lambda} \int_0^x dt \int_0^1 y_{n-1}(x, E/\eta) f(\eta) \frac{d\eta}{\eta}. \tag{2.4}$$

So, we obtain successively

$$y_1(x, E) = G(E) + x/\lambda \int_0^1 G(E/\eta_1) f(\eta_1)\, d\eta_1/\eta_1$$

$$y_n(x, E) = G(E) + \sum_{i=1}^{n} \frac{(x/\lambda)^i}{i!} \int_0^1 \ldots \int_0^1 G\left(\frac{E}{\eta_1 \ldots \eta_i}\right) \cdot \frac{f(\eta_1)}{\eta_1} \ldots \frac{f(\eta_i)}{\eta_i} d\eta_1 \ldots d\eta_i. \tag{2.5}$$

The uniform convergence of the solution (2.5) is ensured if the integral $\int_0^1 f(\eta)\, d\eta/\eta$ exists. So, the nucleon flux at the depth x and at the energy in the interval

E to $E+\mathrm{d}E$ is

$$F(x, E)=\mathrm{e}^{-x/\lambda}\left\{G(E)+\sum_{n=1}^{\infty}\frac{(x/\lambda)^n}{n!}\right.$$
$$\left.\times\int_0^1\cdots\int_0^1 G\left(\frac{E}{\eta_1\ldots\eta_n}\right)\frac{f(\eta_1)\ldots f(\eta_n)}{\eta_1\ldots\eta_n}\,\mathrm{d}\eta_1\ldots\mathrm{d}\eta_n\right\}. \tag{2.6}$$

3. Diffusion equation for secondary hadrons in the atmosphere

The diffusion of the secondary particles m (where m may represent $\pi^{\pm}$, $K^{\pm}$, D^0, etc.) in the atmosphere can be described by the one-dimensional differential equation

$$\frac{\partial M(x, E)}{\partial x}=-M(x, E)\left(\frac{1}{\lambda_m}+\frac{b}{Ex}\right)+P_m^N(x, E)+P_m^m(x, E) \tag{3.1}$$

with the boundary condition

$$M(0, E)=0$$

b is the decay constant of the secondary particle, m, in the atmosphere. $P_m^N(x, E)$ is the rate of production of secondary particles m originated by the nucleon–air nuclei interactions, with energy between E and $E+\mathrm{d}E$ at the depth x. $P_m^m(x, E)$ is the similar rate for m–air nuclei interactions. They are given by the expressions

$$\begin{aligned} P_m^N(x, E)&=\int_{E'_{\min}}^{E'_{\max}}\frac{N(x, E')}{\lambda}f_N(E, E')\,\mathrm{d}E' \\ P_m^m(x, E)&=\int_{E'_{\min}}^{E'_{\max}}\frac{M(x, E')}{\lambda m}f_m(E, E')\,\mathrm{d}E' \end{aligned} \tag{3.2}$$

with λ_m = interaction mean free path of the meson m in the atmosphere. $f_N(E, E')$ and $f_m(E, E')$ are the energy distributions of the secondary particles m originated by the interactions N–air nuclei, and m–air nuclei, respectively.

If we assume the scaling hypothesis, the equation (3.1), with the expression (3.2), takes the form

$$\frac{\partial M(x, E)}{\partial x}=-M(x, E)\left(\frac{1}{\lambda}+\frac{b}{Ex}\right)+\int_0^1\frac{N(x, E/\eta)}{\lambda_N}f_N(\eta)\frac{\mathrm{d}\eta}{\eta}$$
$$+\int_0^1\frac{M(x, E/\eta)}{\lambda_m}f_m(\eta)\,\mathrm{d}\eta/\eta. \tag{3.3}$$

Let us put

$$M(x, E)=x^{-b/E}y(x, E) \tag{3.4}$$

and define the operator $\hat{A}$:

$$\hat{A}M(x, E)=\int_0^1 M(x, E/\eta)f_m(\eta)\,\mathrm{d}\eta/\eta. \tag{3.5}$$

So, the equation (3.3) and the respective initial condition become

$$\frac{\partial y(x, E)}{\partial x} = -(1-\hat{A})\frac{y(x, E)}{\lambda_m} + x^{b/E}\int_0^1 M(x, E/\eta) f_N(\eta)\frac{d\eta}{\eta} \tag{3.6}$$

and

$$y(0, E) = 0.$$

The term $\hat{A}y(x, E)$ is unknown, and to solve the equation (3.6) we will make the following successive approximations. Initially we obtain the approximation of zero order, $y_0(x, E)$, where we do not include the second generation of secondary particles m. After this we put, in the equation (3.6), the term $\hat{A}y_0(x, E)$ at the place of the exact term $\hat{A}y(x, E)$. We obtain, then, the first estimate for the contribution of the second generation of m-particles to the total flux.

Following up, we make the successive assessment of the contribution of the 3rd, 4th . . . nth generations to the total flux.

This procedure is represented by the following recurrence equations

$$\begin{aligned} \frac{\partial y_0}{\partial x} &= -\frac{y_0(x, E)}{\lambda_m} + P_0(x, E) \\ \frac{\partial y_n}{\partial x} &= -\frac{y_n(x, E)}{\lambda_m} + P_n(x, E) \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} P_0(x, E) &= x^{b/E} P_m^N(x, E) \\ P_n(x, E) &= x^{b/E} P_m^N(x, E) + \frac{\hat{A}y_{n-1}(x, E)}{\lambda_m}. \end{aligned} \tag{3.8}$$

The solutions to the system of linear equations must satisfy the following boundary condition

$$y_n(0, E) = 0 \qquad n = 0, 1, 2, \ldots .$$

The functions $P_n(x, E)$ and $F_n(x, E)$, $(n=0, 1, 2 \ldots)$ must be continuous in the domain $\xi = [0 \leqslant x \leqslant X; E_{\min} \leqslant E \leqslant E_{\max}]$, with $E_{\min} > 0$ $E_{\max} > E_{\min}$ and $x > 0$. This is satisfied when:

(a) $G(E)$ is a continuous and limited function in the interval $I = [E_{\min}, \infty)$, $E_{\min} > 0$. These functions are positive because they represent the primary energy spectrum,

(b) $f_N(\eta)$ and $f_m(\eta)$ are continuous and non-negative functions in the interval $0 \leqslant \eta \leqslant 1$; and

(c) the integrals $\int_0^1 f_N(\eta)\, d\eta/\eta$ and $\int_0^1 f_m(\eta)\, d\eta/\eta$ exist.

If these conditions are satisfied, the unique and closed solution of the systems (3.7) and (3.8) is

$$y_n(x, E) = \hat{B}P_n(x, E) = \int_0^x e^{((x-t)/\lambda_N)} P_n(t, E)\, dt \tag{3.9}$$

Then

$$y_0(x, E) = \hat{B}P_0(x, E) = \hat{B}x^{b/E}P_m^N(x, E)$$
$$y_n(x, E) = \hat{B}(1 + \hat{A}\hat{B} + \ldots + (\hat{A}\hat{B})^n)x^{b/E}P_m^N(x, E). \quad (3.10)$$

As the order of integration here is irrelevant, the nth approximation, $y_n(x, E)$, can be put in the following form,

$$y_n(x, E) = \hat{B}(1 + \hat{A}\hat{B} + \ldots + \hat{A}^n\hat{B}^n) \times^{b/E} P_m^N(x, E) \quad (3.11)$$

and if the conditions a, b and c cited above are satisfied the series converges uniformly to the function

$$y(x, E) = \sum_{n=0}^{\infty} y_n(x, E).$$

The differential flux of the secondary particles m at depth x and energy between E and $E + \mathrm{d}E$ is

$$M(x, E) = \sum_{i=0}^{\infty} \hat{A}^i\hat{B}^{i+1}x^{b/E}P_m^N(x, E). \quad (3.12)$$

4. Differential muon and neutrino vertical fluxes

The production spectrum of muons and neutrinos, d(μ or ν), is given by

$$P_d(E, x) = \sum_n \int_{E^-}^{E^+} \mathrm{d}E' \frac{Bb}{E'x} f_d(E', E)M(x, E') \quad (4.1)$$

where $M(x, E')$ is the differential flux of m-mesons at depth x and energies between E' and $E' + \mathrm{d}E'$; E', E are, respectively, the energies of the primary mesons m and of the secondary particles d, B is the branching ratio of the mesons m, $f_d(E', E)$ is the inclusive spectrum of secondaries d from decay of particles m with energy E. The values E^-, E^+ and the functions $f_d(E', E)$ are obtained from relativistic kinematic considerations of two and three-body decays [3].

The differential fluxes of leptons d are derived from the production spectrum (4.1);

$$D(E, x) = \int_0^x P_d(E, x)W(t, x, E)\,\mathrm{d}t \quad (4.2)$$

where $W(t, x, E)$ is the probability that a lepton d with energy E produced in a depth t will survive until the depth x.

5. Particular case

If the primary energy spectrum of nucleons is $N(0, E) = N_0E^{-(\gamma+1)}$, the solutions (2.7) and (3.12) will take the simplified expressions as follows.

5.1. Differential nucleon flux

The multiple integrals

$$I=\int_0^1 \dots \int_0^1 N_0\left(\frac{E}{\eta_1\eta_2\dots\eta_n}\right)^{-(\gamma+1)} f(\eta_1)\dots f(\eta_n)\,\frac{d\eta_1\dots d\eta_n}{\eta_1\dots\eta_n}$$

which appear in the solution (2.7), result in $NoE^{-(\gamma+1)}(C_{NN})^n$, where

$$C_{NN}=\int_0^1 \eta^\gamma f(\eta)\,d\eta.$$

The nucleon flux, then, becomes

$$N(x, E)=e^{-x/\lambda_N}\sum_{n=0}^{\infty}\frac{(x/\lambda_N)^n}{n!}(C_{NN})^n N_0 E^{-(\gamma+1)}$$

which is equivalent to the usual expression

$$N(x, E)=N_0E^{-(\gamma+1)}\,e^{-x/L}$$

where $L=\lambda/(1-C_{NN})$ is the absorption mean free path of nucleons in the atmosphere.

5.2. Differential flux of secondary particles

The production rate of secondaries m from the nucleon-air nuclei interaction, is

$$P_m^N(x, E)=\frac{N(x, E)}{\lambda}C_{Nm} \tag{5.1}$$

where

$$C_{Nm}=\int_0^1 \eta^\gamma f_N(\eta)\,d\eta. \tag{5.2}$$

Applying the operator $\hat{B}$, $(n+1)$ times in the expression $x^{b/E}P_m^N(x, E)$, making the substitution $t_1=x-\tau$, and using the properties of iteratives integrals (4), we obtain

$$\hat{B}^{n+1}x^{b/E}P_m^N(x, E)=e^{-x/\lambda}\int_0^x \exp\left[\left(\frac{1}{\lambda}-\frac{1}{\lambda_m}\right)\tau\right]\frac{\tau^n}{n!}(x-\tau)^{b/E}N_0E^{-(\gamma+1)}\frac{C_{Nm}}{\lambda}\,d\tau. \tag{5.3}$$

Applying n times the operator $\hat{A}$ in the last equation, and defining the integrals

$$C_{mm}=\int_0^1 \eta^\gamma f_m(\eta)\,d\eta$$

we obtain for the differential flux of secondary particles

$$M(x, E)=\frac{N_0E^{-(\gamma+1)}C_{Nm}}{\lambda}e^{-x/L_m}\int_0^x\left(\frac{t_1}{x}\right)^{b/E}\exp\left[-\left(\frac{1}{L}-\frac{1}{L_m}\right)t_1\right]dt_1 \tag{5.4}$$

where L_m is the absorption mean free path of the secondaries m in the atmosphere.

If $E \gg b$, the expression $(t_1/x)^{b/E}$ is approximately 1 and the solution (5.6) takes the well known form

$$M(x, E) = \frac{N_0 E^{-(\gamma+1)} C_{Nm}}{\lambda_m} \frac{e^{-x/L_m} - e^{-x/L}}{1/L - 1/L_m}. \tag{5.5}$$

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